

Duality Symmetries and G^{+++} Theories

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We show that the non-linear realisations of all the very extended algebras G^{+++} , except the B and C series which we do not consider, contain fields corresponding to all possible duality symmetries of the on-shell degrees of freedom of these theories. This result also holds for G_2^{+++} and we argue that the non-linear realisation of this algebra accounts precisely for the form fields present in the corresponding supersymmetric theory. We also find a simple necessary condition for the roots to belong to a G^{+++} algebra.

The IIA [1] and IIB [2,3,4] supergravity theories, by virtue of the large amount of supersymmetry that they possess, encode all the low energy effects of their corresponding string theories, while the eleven dimensional supergravity theory [5] is thought to be the low energy effective action for an as yet undefined theory called M theory. Given that there exists no complete formulation of string theory these supergravity theories have proved invaluable in our understanding of string theory and extensions of it to include branes. The result [6] that these supergravity theories can be formulated as non-linear realisations led to the conjecture [7] that the non-linear realisation of the generalised Kac-Moody algebra E_{11} was an extension of eleven dimensional supergravity. Indeed this algebra, when decomposed with respect to its A_{10} , or $SL(11)$, algebra associated with eleven dimensional gravity, has lowest level positive root generators which are in one to one correspondence with the graviton, the three and six form gauge fields, while the next generator corresponds to the dual graviton [7]. Non-linear realisations of E_{11} can also be used to describe the maximal ten dimensional supergravity theories, but in this case the adjoint representation is decomposed into representations of A_9 , or $SL(10)$, associated with ten dimensional gravity. There are only two such A_9 subalgebras and the two different choices were found to lead to non-linear realisations which at low levels are the IIA and IIB supergravity theories in references [7] and [8] respectively. It is striking to examine tables of the generators [9] listed in terms of increasing level and see how the generators associated with the field content of the IIA and IIB supergravity theories occupy precisely all the lower levels, before an infinite sea of generators whose physical significance was unknown at the time the tables of reference [9] were constructed. The fact that the three maximal supergravity theories in ten and eleven dimensions could be formulated in terms of a single E_{11} theory encouraged the belief [7] that this algebra might be a symmetry of the underlying M theory.

Amongst the set of E_{11} generators appropriate to the IIA theory is one with nine antisymmetrised indices which in the non-linear realisation leads to a field with the same index structure [9]. This nine form generator occurs in the table of IIA generators at a place which is amongst the generators associated with the fields of the IIA supergravity theory. A non-trivial value for this field is known to lead to the massive IIA supergravity theory [10] and as a result it was realised [9] that E_{11} can incorporate the massive IIA theory.

However, there is a one to one correspondence between the fields of the non-linear realisations of E_{11} appropriate to the eleven dimensional, IIA and IIB theories and using this correspondence one finds that the nine form of the IIA theory corresponds to a field with the index structure $A^{a,b,c_1\dots c_{10}}$ which occurs at a level which is beyond those of the supergravity fields of eleven dimensional supergravity theory [11]. As such E_{11} provides an eleven dimensional origin of the massive IIA theory which involves one of the higher level fields and in this way the physical interpretation of at least one of the higher level fields in the eleven dimensional E_{11} theory became apparent.

For quite some time the significance of the higher fields in the non-linear realisation, with the exception of the one field just mentioned above, was unclear. However, it was shown that the quadruplet and doublet space-filling forms of the IIB theory which were known [9] to be present in the E_{11} formulation were found from a rather different perspective. Remarkably it was shown [12] that if one included the dual fields corresponding

to all the physical fields of the IIB supergravity theory then the supersymmetry algebra could be closed in the presence of a set of ten forms which were precisely the fields that were predicted by E_{11} . Furthermore, it was shown that there was a precise matching of coefficients for the gauge algebras of all these forms found on the one hand from closing the supersymmetry algebra and on the other hand from the E_{11} algebra [13]. A similar story applies to the closure of the IIA supergravity algebra [14] and the space filling forms predicted from E_{11} [9].

Recently considerable numbers of higher level E_{11} fields were shown to have a physics meaning. First it was shown [15] that an infinite class of the higher fields were just those required to realise all possible duality symmetries of the basic on-shell degrees of freedom of the theory. More recently all the form fields, that is those fields whose indices are totally antisymmetrised, resulting from the dimensional reduction of the eleven dimensional non-linear realisation were calculated [16]. It emerged that the formulation of the maximal supergravities in the lower dimension D which arose from E_{11} was democratic meaning that the physical degrees of freedom of the theory were described by two fields whose field strengths were related by Hodge duality, except in the case of self-dual fields. Furthermore, the rank $D - 1$ forms that arose could be used to classify all possible deformations of the maximal supergravity theories that arise from a Lagrangian formulation and we found that the results were in complete agreement with the known maximal gauged supergravity theories. The latter have been found over many years by considering the deformations of the massless maximal supergravity theory in the dimension of interest (see for example reference [17] and references therein). This result also provided an eleven dimensional origin for all the gauged, or massive, maximal supergravity theories. Reference [16] also found all the space-filling forms which are important for the consistency of orientifold models [18]. The results of reference [16] were also found in reference [19] by analysing the E_{11} theory directly in the dimension of interest.

Arguments similar to those advocated for eleven dimensional supergravity in [7] were applied to the effective action of the closed bosonic string D dimensions [7], to gravity in D dimensions [20], and the type I supergravity theory [21] and the underlying Kac-Moody algebras were identified. It was realised that the algebras that arose in all these theories were of a special kind and were called very extended Kac-Moody algebras [22]. Indeed, for any finite dimensional semi-simple Lie algebra \mathcal{G} one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac-Moody algebra denoted \mathcal{G}^{+++} . In this notation E_{11} is written as E_8^{+++} . The Kac-Moody algebras that were conjectured to underlie the closed bosonic string, gravity and type I supergravity being D_{D-2}^{+++} [7], A_{D-3}^{+++} [20], and D_8^{+++} [21] respectively.

The ideas in [7,20,21,22] were generalised in references [23,24] to consider the non-linear realisation of any \mathcal{G}^{+++} algebra. For each very extended algebra \mathcal{G}^{+++} one can find an A_{D-1} sub-algebra that is associated with a set of $D - 1$ nodes which form a line in the \mathcal{G}^{+++} Dynkin diagram. This line is called the gravity line and it must start with the very extended node. In the non-linear realisation the A_{D-1} sub-algebra is associated with the gravity sector of the theory and the resulting theory lives in D dimensions. In general there is more than one possible gravity line, or A_{D-1} sub-algebra. By analysing \mathcal{G}^{+++} with respect to the A_{D-1} sub-algebra it was shown [9] that the low level fields present in

the non-linear realisation of \mathcal{G}^{+++} contained gravity, form gauge fields and in some cases scalars which was the result anticipated by the above conjectures. This was also consistent with the oxidation points [24] of all the three dimensional theories which possessed coset symmetries.

In this paper we investigate if the \mathcal{G}^{+++} algebras have some of the same features described above for the E_8^{+++} algebra. In particular we will see that if the gravity sub-algebra of \mathcal{G}^{+++} being considered is A_{D-1} then we can define dual generators to be those that have no blocks of anti-symmetrised D or $D - 1$ indices. We will then find all dual generators and show that they are only those corresponding to the on-shell degrees of freedom of the theory together with those fields whose field strengths are related to these by Hodge duality, as well as generators that have the same index structure as these fields, but are decorated by any number of blocks of $D - 2$ indices. We will show this result for all \mathcal{G}^{+++} algebras with the exception of the B and C algebras which we do not consider. This result implies that the non-linear realisation encodes all possible duality symmetries of the on-shell degrees of freedom of these theories.

We also derive the form fields of the \mathcal{G}_2^{+++} theory in five, four and three dimensions and discuss the possible resulting deformations of the theory. The results are completely compatible with the know literature on the subject. In two separate appendices, we also derive the form fields arising in any dimension from 8 to 4 for the E_6^{+++} theory and from 6 to 4 for the F_4^{+++} theory, while a third appendix is devoted to a discussion of the G_2^{+++} case.

1 E_8^{+++}

Any generator in a Kac-Moody algebra can by definition be written as the multiple commutators of the Chevalley generators (H_a, E_a and F_a) which are subject to the Serre relations. However for no general Kac-Moody algebra is even a complete listing of the generators known. A Lorentzian Kac-Moody algebra [22] is one where the deletion of at least one node in the Dynkin Diagram will leave a finite dimensional algebra and at most one affine algebra. These algebras are more tractable, in that their properties can be analysed in terms of the well understood algebra that remains after the deletion of the preferred node in the Dynkin diagram. In particular, one can analyse the adjoint representation of the Kac-Moody algebra in terms of the representations of the algebra that remains. The level of a given generator of the Kac-Moody algebra is defined to be the number of times the generator corresponding to the deleted node occurs in the multiple commutator in which the generator arises. Often one can delete a node such that the algebra that remains is A_n and if this is not the case one can delete another node to find an A_n algebra. In this section we will review how the determination of the properties of the Kac-Moody algebra are determined in terms of the A_n sub-algebra [22,25,26] and recall how it works in detail for the case of E_{11} [26]. We will then recover the dual generators of reference [15], but also give some new insights into how to determine the remaining generators of the algebra.

For an algebra where only one node, labelled by c , needs to be deleted to find an A_n gravity subalgebra, the simple roots of the Kac-Moody algebra can be written in terms

of those of A_n , that is α_i , $i = 1, \dots, n$, and the simple root corresponding to the deleted node which can be written as [22]

$$\alpha_c = x - \nu \quad (1.1)$$

where ν is given by

$$\nu = - \sum_i A_{ci} \lambda_i, \quad (1.2)$$

x is a vector orthogonal to the A_n weight lattice and the λ_i are the fundamental weights of the A_n subalgebra. Indices a, b, \dots run over the rank of the full Kac-Moody algebra, while i, j, \dots over the rank of the A_n sub-algebra. We note that the simple roots do indeed replicate the Cartan matrix A_{ab} of the Kac-Moody algebra, which if this is symmetric are given by $(\alpha_a, \alpha_b) = A_{ab}$. Here we used that $(\alpha_i, \lambda_j) = \delta_{ij}$ for the A_n sub-algebra as indeed is the case for all simply laced finite dimensional semi-simple Lie algebras. The quantity x^2 is determined by demanding that α_c^2 have the correct value, which for the case of symmetric Cartan matrix is just $\alpha_c^2 = 2$.

Any root of the Kac-Moody algebra can be written, using equation (1.2), in the form

$$\alpha = \sum_i n_i \alpha_i + l \alpha_c = lx - \Lambda, \quad \text{where } \Lambda = \nu - \sum_i n_i \alpha_i. \quad (1.3)$$

We recognise l as the level. We see that Λ belongs to the weight lattice of the A_n subalgebra. If a representation of A_n , with highest weight $\sum_i p_i \lambda_i$, where p_j are the Dynkin indices, occurs then this highest weight must occur as one of the possible Λ 's as the roots of the Kac-Moody algebra vary. As such, a necessary condition for the adjoint representation of the generalised Kac-Moody algebra to contain the highest weight representation of A_n with Dynkin indices p_j [22, 25,26] is that

$$\sum_j p_j \lambda_j = l \nu - \sum_i n_i \alpha_i. \quad (1.4)$$

Taking the scalar product with λ_k leads to the condition

$$n_k = l \nu \cdot \lambda_k - \sum_j p_j (\lambda_j, \lambda_k). \quad (1.5)$$

We recall that $(\lambda_i, \lambda_j) = A_{ij}^{-1}$ for any simply laced algebra, where A_{ij}^{-1} is the inverse Cartan matrix which is positive definite for a finite dimensional semi-simple Lie algebra. The inverse Cartan matrix of the A_{D-1} algebra is given by

$$A_{ij}^{-1} = \begin{cases} \frac{i(D-j)}{D}, & i \leq j \\ \frac{j(D-i)}{D}, & j \leq i \end{cases} \quad (1.6)$$

In equation (1.5) the integers l, n_k, p_j must be positive and so this places a necessary, but not sufficient, condition on the possible A_n representations contained in the adjoint representation of the Kac-Moody algebra at level l .

Taking the scalar product of the expression for α of equation (1.3) we find that

$$\alpha^2 = l^2 x^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j) p_j = 2, 0, -2, \quad (1.7)$$

We have used the fact that the lengths of the roots of a symmetric Kac-Moody algebra are constrained to take the values $2, 0, -2, \dots$ [27].

Thus we find a second necessary, but not sufficient, constraint on the A_n representations contained in the adjoint representation of the Kac-Moody algebra. In fact the two constraints of equations (1.5) and (1.7) are not as strong as imposing the Serre relations on the multiple commutators, although almost all the solutions they possess are roots that actually appear in the Kac-Moody algebra.

Let us explain this procedure for the case of E_{11} [26] whose Dynkin diagram is given below

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & & 0 & 11 \\ & & & & & & & & & & & | \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 \end{array}$$

Deleting the node 11 leaves an A_{10} subalgebra. The simple roots of E_{11} are those of A_{10} , i.e. α_i , $i = 1, \dots, 10$, as well as the simple root of the deleted node which is given by

$$\alpha_{11} = x - \lambda_8, \quad x^2 = -\frac{2}{11} \quad (1.8)$$

The general root of E_{11} has the form

$$\alpha = l\alpha_{11} + \sum_{i=1}^{10} n_i \alpha_i = lx - \Lambda \quad (1.9)$$

where $\Lambda = l\lambda_8 - \sum_{i=1}^{10} n_i \alpha_i$. This equation effectively rewrites the adjoint representation of E_{11} in terms of representations of A_{10} . A necessary condition for a representation of A_{10} to occur is that the set of all Λ 's contain the highest weight of the representation in question, that is $\sum_i p_i \lambda_i$, where the p_i are the Dynkin indices. As such, we can set $\Lambda = \sum_i p_i \lambda_i$ and taking the scalar product with λ_j we find that

$$\sum_i p_i \lambda_i \cdot \lambda_j - l\lambda_8 \cdot \lambda_j = -n_j \quad (1.10)$$

While the square of the corresponding E_{11} root is given by equation (1.7), that is

$$\alpha^2 = -\frac{2}{11}x^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j) p_j = 2, 0, -2, \dots \quad (1.11)$$

Any generator is found by taking the multiple commutator of the Chevalley generators and the level of a generator is the number of times the Chevalley generator corresponding

to node 11 appears in this multiple commutator. The Chevalley generator associated with node 11 has three A_{10} indices, hence it adds three indices every time it appears in the multiple commutator. Therefore any level l generator can be written with $3l$ indices. We note that the remaining Chevalley generators are contained in the K^a_b generators of A_{10} and so these do not change the number of indices. Given a generators with Dynkin index p_i , the index contributes p_j blocks of $11 - j$ anti-symmetrised indices; as a result the total number of indices obeys the relation [15]

$$3l = \sum_i (11 - i)p_i + 11 m \quad (1.12)$$

The last term corresponds to the possible presence of m blocks of fully antisymmetrised rank 11 indices. Such blocks do not transform under the A_{10} subalgebra.

If we substitute the level condition of equation (1.12) into the root length condition of equation (1.11) we find that

$$\begin{aligned} \alpha^2 &= \frac{1}{9} \sum_{j=1}^{10} (11 - j)(j - 2)p_j^2 + \frac{2}{9} \sum_{i < j} (11 - j)(i - 2)p_i p_j - \frac{4}{9} m \sum_i (11 - i)p_i - \frac{2 \cdot 11}{9} m^2 \\ &= 2, 0, -2, \dots \end{aligned} \quad (1.13)$$

Substituting the level condition of equation (1.12) into the root condition of equation (1.10) we find that the factors of $\frac{1}{11}$ disappear and the values of n_i are given by

$$\begin{aligned} n_j &= \sum_{i, i < j} (j - i)p_i + jm, \quad j = 1, \dots, 8, \quad n_9 = \frac{2}{3} \left(\sum_i (8 - i)p_i + 8m \right) + p_{10}, \\ n_{10} &= \frac{1}{3} \left(\sum_i (8 - i)p_i + 8m \right) \end{aligned} \quad (1.14)$$

We see that the n_i , $i = 1, \dots, 9$ are positive as they must be. Furthermore, using the level condition of equation (1.12) one sees that $\sum_j (8 - j)p_j + 8m$ is a multiple of 3 as $3(l - \sum_j p_j - s) = \sum_j (8 - j)p_j + 8m$ and so n_9 and n_{10} are integers which one can also show are positive.

Hence, we find the perhaps surprising result that the root condition of equation (1.10) is automatically satisfied if we use the level matching condition of equation (1.12). We recall that previously one found the possible roots of E_{11} by finding all the solution of equations (1.10) and (1.11). However, now we need only solve the level matching condition of equation (1.12) and the reformulation of the length squared condition of equation (1.13). Clearly, these are much simpler conditions than those of equations (1.10) and (1.11).

In reference [15] the concept of dual generators was introduced; these are generators which possess with no blocks of rank 10 or 11 totally antisymmetrised indices. This means that $m = 0 = p_1$ and so now $3l = \sum_j p_j(11 - j)$. We observe that in the root length condition of equation (1.13) all terms on the right-hand side become positive and that as

p_2 is absent this Dynkin index has no condition placed on it. Taking $p_2 = 0$, the only allowed roots are given by

$$\begin{aligned}\alpha_A &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1), \quad p_8 = 1 \\ \alpha_B &= (0, 0, 0, 0, 0, 1, 2, 3, 2, 1, 2), \quad p_5 = 1 \\ \alpha_C &= (0, 0, 0, 1, 2, 3, 4, 5, 3, 1, 3), \quad p_3 = p_{10} = 1\end{aligned}\tag{1.15}$$

The other p_i s are 0. These correspond in the non-linear realisation to the three form, six form and dual graviton generators respectively and as expected they occur with multiplicity one. The solution for $p_2 = 1$ is given by

$$\gamma = (0, 0, 1, 2, 3, 4, 5, 6, 4, 2, 3), \quad p_2 = 1\tag{1.16}$$

This solution has multiplicity zero and so does not actually occur in the E_{11} algebra. It corresponds to a generator of the form $R^{a_1 \dots a_9}$.

The roots corresponding to all the dual generators can then be written in the form [15]

$$\begin{aligned}\alpha_A(p_2) &= \alpha_A + p_2 \gamma \\ \alpha_B(p_2) &= \alpha_B + p_2 \gamma \\ \alpha_C(p_2) &= \alpha_C + p_2 \gamma \\ \alpha_D(p_2) &= p_2 \gamma\end{aligned}\tag{1.17}$$

They are just found by taking multiple commutators of the multiplicity generator $R^{a_1 \dots a_9}$ with each of the generators corresponding to the roots of equation (1.15).

The new fields found in the non-linear realisation corresponding to the roots of equation (1.17) differ from those of equation (1.15) by blocks of nine indices. However, as the little group of the massless states is $SO(9)$ these fields describe the same on-shell states. As a result, we find that the E_{11} non-linear realisation includes all possible ways of describing the original degree of freedom of the theory and so we may conclude that E_{11} encodes all possible duality transformations [15].

We now apply the above arguments to the other very extended algebras.

2 A_{D-3}^{+++}

It has been conjectured that a suitably extended version of pure gravity in D dimensions can be expressed as a nonlinear realisation of A_{D-3}^{+++} [20]. Analysing this algebra with respect to its A_{D-1} sub-algebra we find at level 0 the A_{D-1} generators K^a_b , corresponding to the graviton, and at level 1 the generator $R^{a_1 \dots a_{D-3}, b}$ corresponding to the dual graviton field. The Dynkin diagram of A_{D-3}^{+++} is given by

$$\begin{array}{ccccccccccc} & & & & & & D & & & & \\ & & & & & & - & - & 0 & - & - \\ & & & & & & & & & & \\ & & & & & & | & & & & | \\ & & & & & & 0 & & & & 0 \\ & & & & & & - & & & & - \\ & & & & & & 0 & \dots & 0 & & 0 \\ & & & & & & 1 & & 2 & & 3 \\ & & & & & & & & & & D & -1\end{array}$$

For simplicity we will assume that $D > 4$. Deleting the node labelled D we find the A_{D-1} gravity subalgebra. The simple roots of A_{D-3}^{+++} can be written as the simple roots α_i , $i = 1, \dots, D-1$ of A_{D-1} as well as the simple root of the deleted node which is given by

$$\alpha_D = x - \lambda_3 - \lambda_{D-1} . \quad (2.1)$$

Given $\alpha^2 = 2$ we find $x^2 = \frac{4}{D} - 2$. As such, a general root of A_{D-3}^{+++} can be written as

$$\alpha = \sum_i n_i \alpha_i + l \alpha_D = lx - \Lambda, \quad \text{where } \Lambda = l(\lambda_3 + \lambda_{D-1}) - \sum_i n_i \alpha_i . \quad (2.2)$$

A highest weight representation of A_{D-1} with Dynkin indices p_i can occur if $\Lambda = \sum_i p_i \lambda_i$ occurs. Taking the scalar product of this equation with λ_k gives an equation for the root components

$$n_k = l((\lambda_3, \lambda_k) + (\lambda_k, \lambda_{D-1})) - \sum_j p_j (\lambda_j, \lambda_k) . \quad (2.3)$$

Squaring the expression for α in (2.2) gives

$$\alpha^2 = -\frac{2(D-2)}{D}l^2 + \sum_{i,j} p_i (\lambda_i, \lambda_j) p_j = 2, 0, -2, \dots \quad (2.4)$$

The Chevalley generator corresponding to the deleted node D has a block of $D-3$ antisymmetrised A_{D-1} indices, and a vector index, i.e. $R^{a_1 \dots a_{D-3}, b}$, giving $D-2$ indices overall. A level l generator, the multiple commutator of which contains the generator corresponding to the deleted node l times, will therefore have $(D-2)l$ A_{D-1} indices in total. Hence we get

$$\sum_j p_j (D-j) + Dm = (D-2)l \quad (2.5)$$

where m is the number of blocks of D antisymmetrised indices. Substituting this expression for l into (2.4) we find that

$$\begin{aligned} \alpha^2 = \frac{1}{(D-2)} & \left(\sum_j p_j^2 (D-j)(j-2) + 2 \sum_{i < j} p_i p_j (D-j)(i-2) \right) \\ & - \frac{4m}{(D-2)} \sum_i (D-i)p_i - \frac{2m^2 D}{(D-2)} = 2, 0, -2, \dots \end{aligned} \quad (2.6)$$

We now show that any generator that satisfies (2.5) automatically satisfies the root condition (2.3), namely that it gives positive integer values for the n_k . Substituting (2.5) into (2.3) we find that the factors of $\frac{1}{D}$ disappear and it gives

$$n_j = \begin{cases} p_{j-1} + jm, & j = 1, 2, 3 \\ \sum_{i, i < j} (j-i)p_i + jm - (j-3)l, & j = 3, \dots, D-1 \end{cases} \quad (2.7)$$

where we have formally taken $p_0 = 0$.

The definition of a dual generator is that it has no blocks D or $D - 1$ totally antisymmetric A_{D-1} indices. This may be written $p_1 = m = 0$ and so the level matching condition becomes

$$\sum_j p_j (D - j) = (D - 2)l, \quad (2.8)$$

while equation (2.6) becomes

$$\alpha^2 = \frac{1}{(D - 2)} \left(\sum_j p_j^2 (D - j)(j - 2) + 2 \sum_{i < j} p_i p_j (D - j)(i - 2) \right) = 2, 0, -2, \dots \quad (2.9)$$

Two things are immediately obvious from this equation; p_2 is unconstrained, and both terms are positive definite, so α^2 can only be 2 or 0. For $\alpha^2 = 0$, both terms must be 0, so only p_2 may be nonzero. Equation (2.5) then implies that $l = p_2$ and equation (2.7) gives the first $D - 1$ coefficients $(n_1 \dots n_{D-1})$ of the root, while the last coefficient of the root is simply the level, $n_D = l = p_2$. Defining γ to correspond to the coefficients $n_j = (0, 0, 1, \dots, 1, \dots, 1)$, gives the roots corresponding to all $\alpha^2 = 0$ dual generators in terms of the free parameter p_2 by

$$\alpha_B(p_2) = p_2 \gamma. \quad (2.10)$$

In fact the generator with $p_2 = 1$ has multiplicity zero and so does not appear in the algebra.

The remaining solutions have $\alpha^2 = 2$. We notice that $p_3 = p_{D-1} = 1$, with $p_2 = 0$ and all other $p_i = 0$ gives a solution with $\alpha^2 = 2$. Using equation (2.7) we find that the corresponding root is given by

$$\alpha_A = (0, 0, \dots, 0, 1), \quad p_3 = 1 = p_{D-1}. \quad (2.11)$$

This corresponds to the generator $R^{a_1 \dots a_{D-3}, b}$ which has multiplicity one and so up to this level the theory contains gravity and the dual graviton [20,9].

Since p_2 does not appear in equation (2.6) given any solution we may find a whole class of solutions that have the same Dynkin indices as before, but with the addition of a p_2 that can be any positive integer. Adding such a p_2 leads to a root also satisfies the level matching condition of equation (2.8) as $p_2 \rightarrow p_2 + 1$ just change $l \rightarrow l + 1$. We recall that equation (2.3) is automatically satisfied if the level matching condition holds. Applying this to the solution of equation (2.11) we have new solutions which can be written in the form

$$\alpha_A(p_2) = \alpha_A + p_2 \gamma \quad (2.12)$$

This line of argument will apply to all the G^{+++} algebras considered in this paper.

In fact all possible dual generators are given in equations (2.10) and (2.12) and summarising we find that all the dual generators present in the A_{D-3}^{+++} algebras may be written as

$$\begin{aligned} \alpha_A(p_2) &= \alpha_A + p_2 \gamma \\ \alpha_B(p_2) &= p_2 \gamma \end{aligned} \quad (2.13)$$

We now explain why there are no other solutions. The contribution from a single p_i to α^2 is given by

Let us suppose that a single p_i is non-zero and takes the value $p_i = r$. The level matching condition of equation (2.8) becomes $(D - 2)l = (D - i)r$ which in turn implies that $(D - 2)(l - r) = -(i - 2)r$. Using this latter condition and that $\alpha^2 = 2$ we find that $2 = rl(i - 2)$ which has no solution that is compatible with the level matching condition.

$$\alpha^2 = 3(i-2) + (j-2) - \frac{(i-j)^2}{(D-2)} . \quad (2.15)$$

3 E_7^{+++}

$$\begin{array}{cccccccccc}
 & & & & & & & & 0 & 10 \\
 & & & & & & & & | & \\
 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\
 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9
 \end{array}$$

In the case of E_7^{+++} node ten is deleted to leave an A_9 subalgebra. The simple root corresponding to node ten can be written as

where λ_6 is a fundamental weight of the A_9 subalgebra. Given that $\alpha_{10}^2 = 2$ we find $x^2 = -2/5$. A general root of E_7^{+++} can be written in terms of the simple roots

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A highest weight representation of A_{D-1} with Dynkin indices p_j can occur if we can find amongst the roots of E_7^{+++} one such that $\Lambda = \sum_j p_j \lambda_j$. Combining this with (3.2) and taking the scalar product with λ_i implies that

$$n_k = l(\lambda_6, \lambda_k) - \sum_j p_j (\lambda_j, \lambda_k) . \quad (3.3)$$

Squaring the root α of (3.2) and applying the knowledge of the known lengths of roots in simply laced Kac-Moody algebras one finds

$$\alpha^2 = -\frac{2}{5}l^2 + \sum_{i,j} p_i (\lambda_i, \lambda_j) p_j = 2, 0, -2, \dots \quad (3.4)$$

The Chevalley generator corresponding to the deleted node 10 has 4 indices and so a generator with level l will have $4l$ A_{D-1} indices. Any block of 10 fully anti-symmetrised indices will not transform under the A_9 sub-algebra. Using m to denote the number of such blocks we may write

$$\sum_{j=2}^9 p_j (10 - j) + 10m = 4l . \quad (3.5)$$

Substituting this into (3.4) gives

$$\begin{aligned} \alpha^2 = \frac{1}{8} \left[\sum_{j=1}^9 p_j^2 (10 - j)(j - 2) + 2 \sum_{i,i < j} p_i p_j (10 - j)(i - 2) \right] \\ - \frac{1}{2}m \sum_{j=1}^9 (10 - j)p_j - \frac{5}{2}m^2 = 2, 0, -2, \dots \end{aligned} \quad (3.6)$$

We now show that any generator that satisfies (3.5) will automatically satisfy the root condition (3.3). In particular, substituting (3.5) into (3.3) we find that the coefficients of the roots are given by

$$\begin{aligned} n_j = \sum_{i < j} p_i (j - i) + jm, \quad j = 1, \dots, 6, \quad n_7 = \frac{3}{4} \left(\sum_{i=1}^9 (6 - i)p_i + 6m \right) + 2p_9 + p_8 , \\ n_8 = \frac{2}{4} \left(\sum_{i=1}^9 (6 - i)p_i + 6m \right) + p_9, \quad n_9 = \frac{1}{4} \left(\sum_{i=1}^9 (6 - i)p_i + 6m \right) . \end{aligned} \quad (3.7)$$

We note that $4l - 4 \sum_i p_i - 4m = \sum_{i=1}^9 (6 - i)p_i + 6m$. Also, the n_j are all integers as required.

Dual generators are defined to be those with no blocks of nine or ten fully antisymmetrised indices. This condition may be written as

$$p_1 = m = 0 . \quad (3.8)$$

Substituting this into (3.6) gives

$$\alpha^2 = \frac{1}{8} \left[\sum_j^9 p_j^2 (10-j)(j-2) + 2 \sum_{i < j}^9 p_i p_j (10-j)(i-2) \right] = 2, 0, -2, \dots \quad (3.9)$$

This equation is independent of p_2 and its terms are positive. Setting $p_2 = 0$ we find that the only solutions are given by

$$\begin{aligned} \alpha_A &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 1), \quad p_6 = 1 \\ \alpha_B &= (0, 0, 0, 1, 2, 3, 2, 1, 0, 2), \quad p_3 = p_9 = 1 \end{aligned} \quad (3.10)$$

where all other $p_i = 0$. Solutions A and B correspond to the generators $R^{a_1 \dots a_4}$ and $R^{a_1 \dots a_7, b}$ respectively. These have multiplicity one. The resulting low level field content in the non-linear realisation is a self dual four form $A_{a_1 \dots a_4}$ and the dual graviton $A_{a_1 \dots a_7, b}$ [9].

Setting $p_2 = 1$ we find the solution

$$\gamma = (0, 0, 1, 2, 3, 4, 3, 2, 1, 2). \quad (3.11)$$

The generator corresponding to this root has 0 multiplicity and so it does not appear in the algebra.

The roots corresponding to all possible dual generators in E_7^{+++} can then be written in the form

$$\begin{aligned} \alpha_A(p_2) &= \alpha_A + p_2 \gamma \\ \alpha_B(p_2) &= \alpha_B + p_2 \gamma \\ \alpha_C(p_2) &= p_2 \gamma \end{aligned} \quad (3.12)$$

As in all the other cases, this corresponds to the presence of all possible duality transformations.

4 D_{D-2}^{+++}

The nonlinear realisation of the very extended D_{D-2}^{+++} algebras was conjectured as a symmetry of a suitably extended low energy effective action of the bosonic string in D dimensions[20]. The Dynkin diagram of D_{D-2}^{+++} is given by

$$\begin{array}{cccccccccccccccc} & & & & & 0 & D+1 & & & & & & 0 & D & & & \\ & & & & & | & & & & & & & | & & & & \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 & . & . & . & 0 & - & 0 & - & 0 \\ 1 & & 2 & & 3 & & 4 & & 5 & & & & D-3 & & D-2 & & D-1 \end{array}$$

For simplicity we consider $D \geq 7$; the Dynkin diagram has a slightly different structure for smaller D . For the D_{D-2}^{+++} series of algebras two nodes must be deleted to give an A_{D-1}

algebra. The simple roots of D_{D-2}^{+++} are the simple roots of the D_D algebra and the simple root of the deleted node $D + 1$ which may be written as

$$\alpha_{D+1} = y - l_4 \quad (4.1)$$

where l_4 denotes a fundamental weight of the D_D algebra. Next we delete node D to find an A_{D-1} gravity sub-algebra. The simple roots of D_{D-2}^{+++} are now given by the simple roots of the A_{D-1} sub-algebra, the simple root of node D , which we may write as

$$\alpha_D = x - \lambda_{D-2} \quad (4.2)$$

and the simple root of equation (4.1). However, we may express the fundamental weight l_4 of D_D in terms of the fundamental weights of A_{D-1} by [22]

$$l_4 = \lambda_4 + \frac{x}{x^2}(\lambda_{D-2}, \lambda_4) \quad (4.3)$$

Noting that $\alpha_{D+1}^2 = \alpha_D^2 = 2$ we find that $x^2 = \frac{4}{D}, y^2 = -2$.

A general root of D_{D-2}^{+++} can be written

$$\alpha = \sum_i n_i \alpha_i + l_x \alpha_D + l_y \alpha_{D+1} = l_y y + x(l_x - 2l_y) - \Lambda, \text{ where}$$

$$\Lambda = l_y \lambda_4 + l_x \lambda_{D-2} - \sum_i n_i \alpha_i \quad (4.4)$$

where there are now two levels, l_x and l_y which refer to the deleted nodes D and $D + 1$ respectively. A highest weight representation of A_{D-1} with Dynkin indices p_i can occur if the roots of the Kac-Moody algebra D_{D-2}^{+++} include the case when $\Lambda = \sum_i p_i \lambda_i$. Taking the scalar product of both sides with λ_k gives

$$n_k = l_y(\lambda_4, \lambda_k) + l_x(\lambda_{D-2}, \lambda_k) - \sum_j p_j(\lambda_j, \lambda_k) \quad (4.5)$$

The square of the corresponding root is given by

$$\alpha^2 = -2l_y^2 + \frac{4}{D}(l_x - 2l_y)^2 + \sum_{i,j} p_i(\lambda_i, \lambda_j)p_j = 2, 0, -2, \dots \quad (4.6)$$

where α^2 has been constrained to take the values given above as it belongs to a Kac-Moody algebra [27].

The Chevalley generator corresponding to node D has 2 A_{D-1} indices, and the Chevalley generator corresponding to node $D + 1$ has $(D - 4)$ indices. By definition the multiple commutator of a generator with level (l_x, l_y) contains the Chevalley generator corresponding to node D l_x times, adding $2l_x$ indices, and the Chevalley generator corresponding to

node $D + 1$ l_y times, adding $(D - 4)l_y$ indices. In total this gives $2l_x + (D - 4)l_y$ indices. Using m to denote the number of rank D index blocks, we may write

$$\sum_j p_j(D - j) + Dm = (D - 4)l_y + 2l_x . \quad (4.7)$$

Substituting this into (4.6) gives

$$\begin{aligned} \alpha^2 = \frac{1}{D - 2} \left[4d^2 + \sum_i p_i^2(D - 2)(D - i) + 2 \sum_{i < j} p_i p_j(D - j)(i - 2) \right] \\ - \frac{4m}{(D - 2)} \sum_i p_i(D - i) - \frac{2m^2}{(D - 2)} = 2, 0, -2, \dots \end{aligned} \quad (4.8)$$

In deriving this equation we have used the identity

$$-Dl_y^2 + 2(l_x - 2l_y)^2 = \frac{1}{D - 2} \left[2D(l_x - l_y)^2 - ((D - 4)l_y + 2l_x)^2 \right] \quad (4.9)$$

where $d = l_x - l_y$.

In fact, any solution that satisfies equation (4.7) will automatically satisfy the root condition of equation (4.50) and one finds that

$$\begin{aligned} n_j = \begin{cases} \sum_{i, i < j} p_i(j - i) + mj, & j = 1, 2, 3, 4 \\ \sum_{i, i < j} p_i(j - i) + mj + l_y(j - 4), & j = 4, \dots, D - 1 \end{cases} \\ n_{D-1} = l_y + l_x - \sum_i p_i - m \end{aligned} \quad (4.10)$$

Dual generators are defined to be those with no blocks of D or $D - 1$ fully anti-symmetrised indices. This may be written as $p_1 = m = 0$ which when substituting this into equation (4.8) gives

$$\sum_j p_j(D - j) = (D - 4)l_y + 2l_x . \quad (4.11)$$

Equation (4.8) then becomes

$$\alpha^2 = \frac{1}{D - 2} \left[4d^2 + \sum_j p_j^2(D - 2)(j - 2) + 2 \sum_{i < j} p_i p_j(D - j)(i - 2) \right] = 2, 0, -2, \dots \quad (4.12)$$

This equation is independent of p_2 and all terms in the middle equation are positive definite. The general solution to this equation can be found following similar arguments to that deployed for the case of A_{D-3}^{+++} , in particular below equation (2.14).

Setting $p_2 = 0$ we find the following roots

$$\begin{aligned}\alpha_A &= (0, \dots, 0, 1, 0), \quad p_{D-2} = 1, \\ \alpha_B &= (0, \dots, 0, 1), \quad p_4 = 1, \\ \alpha_C &= (0, 0, 0, 1, \dots, 1, 0, 1, 1), \quad p_3 = p_{D-1}.\end{aligned}\tag{4.13}$$

All other p_i are 0. The generators corresponding to the roots A, B and C are $R^{a_1 a_2}$, $R^{a_1 \dots a_{D-4}}$ and $R^{a_1 \dots a_{D-3}, b}$ respectively. These all have multiplicity one.

Setting $p_2 = 1$ we also find the solution

$$\gamma = (0, 0, 1, 2, \dots, 2, 1, 1, 1)\tag{4.14}$$

corresponding to the generator $R^{a_1 \dots a_{D-2}}$ which has multiplicity one.

As a result, the non-linear realisation contains at low levels the fields of gravity and a dilaton, as the rank of D_{D-2}^{+++} is one greater than that D the dimension of space-time, a two form $A_{a_1 a_2}$, its dual $A_{a_1 \dots a_{D-4}}$, the dual graviton $A_{a_1 \dots a_{D-3}, b}$ and the field $A_{a_1 \dots a_{D-2}}$, which is the dual of the dilaton. These are the on-shell states of the effective action of the bosonic string generalised to D dimensions [20,9].

The roots corresponding to all dual generators may be written in the form

$$\begin{aligned}\alpha_A(p_2) &= \alpha_A + p_2 \gamma \\ \alpha_B(p_2) &= \alpha_B + p_2 \gamma \\ \alpha_C(p_2) &= \alpha_C + p_2 \gamma \\ \alpha_D(p_2) &= p_2 \gamma\end{aligned}\tag{4.15}$$

5 E_6^{+++}

At low levels the nonlinear realisation of the very extended E_6 algebra has precisely the field content [9] of the oxidation endpoint of the non-supersymmetric E_6 coset theory described in [28]. The Dynkin Diagram of E_6^{+++} is given by

$$\begin{array}{cccccccccccc} & & & & & & & 0 & 9 & & & \\ & & & & & & & | & & & & \\ & & & & & & & 0 & 8 & & & \\ & & & & & & & | & & & & \\ 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 & - & 0 \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7\end{array}$$

As usual nodes one, two and three are the very extended, over extended and affine nodes. In the case of E_6^{+++} node 8 is deleted leaving the algebra $A_7 \otimes A_1$. This decomposition is similar to the decomposition of E_{11} which is appropriate to IIB supergravity [8,9]. The roots of E_6^{+++} can be written as the roots of A_7 , $\alpha_i, i = 1 \dots 7$, the root of the A_1 algebra β , and the deleted root α_8 , which may be written

$$\alpha_8 = x - \lambda_5 - \mu\tag{5.1}$$

where λ_5 is a fundamental weight of the A_7 algebra and μ is the fundamental weight of the A_1 subalgebra. Knowing $\alpha_8^2 = 2$, we find that $x^2 = -\frac{3}{8}$. A general root of E_6^{+++} may be written as

$$\alpha = \sum_{i=1}^7 n_i \alpha_i + l \alpha_8 + r \beta = lx - \Lambda \quad \text{where} \quad \Lambda = l \lambda_5 + l \mu - \sum_{i=1}^7 n_i \alpha_i - r \beta . \quad (5.2)$$

A highest weight representation of $A_7 \otimes A_1$ with Dynkin indices p_i, q can occur if the E_6^{+++} include the possibility $\Lambda = \sum_i p_i \lambda_i + q \mu$. Dotting this with λ_k and μ in turn gives the pair of equations

$$\begin{aligned} n_k &= l(\lambda_5, \lambda_k) - \sum_{j=1}^7 p_j (\lambda_j, \lambda_k) \\ r &= \frac{l - q}{2} \end{aligned} \quad (5.3)$$

Note that $(\mu, \mu) = 1/2$, and $(\beta, \mu) = 1$. Squaring the expression for α in (5.2) gives

$$\alpha^2 = -3/8 l^2 + \sum_{i,j} (\lambda_i, \lambda_j) p_i p_j + q^2/2 . \quad (5.4)$$

The Chevalley generator corresponding to node 8 has three A_7 indices. As a result, a level l generator, the multiple commutator for which by definition contains the Chevalley generator corresponding to node 8 l times, will have $3l$ A_7 indices in total. We note that the Chevalley generator corresponding to node 9 has no A_7 indices, so it does not contribute to the index count. As a result

$$\sum_j p_j (8 - j) + 8m = 3l \quad (5.5)$$

where m denotes the number of blocks of 8 fully antisymmetrised indices.

Substituting for l from (5.5) into (5.4) gives

$$\begin{aligned} \alpha^2 &= \frac{1}{6} \left(\sum_j p_j^2 (8 - j)(j - 2) + 2 \sum_{i < j} p_i p_j (8 - j)(i - 2) - 16m^2 - 4m \sum_j p_j (8 - j) \right) + \frac{1}{2} q^2 \\ &= 2, 0, -2, \dots \end{aligned} \quad (5.6)$$

We now show that any generator that satisfies (5.5) automatically satisfies the root condition (5.4). Substituting (5.5) into (5.4) we find

$$n_j = \begin{cases} \sum_{i, i < j} p_i (j - i) + jm, & j \leq 5 \\ n_6 = \frac{2}{3} (\sum_i (5 - i) p_i + 5m) + p_7 \\ n_7 = \frac{1}{3} (\sum_i (5 - i) p_i + 5m) \end{cases} \quad (5.7)$$

We note that $3l - 3 \sum_i p_i - 3m = (\sum_i (5 - i)p_i + 5m)$ and the n_j are positive integers as required.

Dual generators are defined to be those with no blocks of 7 or 8 fully antisymmetrised indices. This may be written as

$$p_1 = m = 0 . \quad (5.8)$$

Substituting these into (5.6) gives

$$\alpha^2 = \frac{q^2}{2} + \frac{1}{6} \sum_j p_j^2 (8 - j)(j - 2) + \frac{1}{3} \sum_{i < j} p_i p_j (8 - j)(i - 2) = 2, 0, -2, \dots \quad (5.9)$$

The middle terms in this equation are positive definite and p_2 is undetermined. Setting $p_2 = 0$ gives the solutions

$$\begin{aligned} \alpha_A &= (0, 0, 0, 0, 0, 0, 0, 1, 0), \quad q = 1, p_5 = 1 \\ \alpha_B &= (0, 0, 0, 1, 2, 1, 0, 2, 1), \quad q = 0, p_3 = p_7 = 1 \\ \alpha_C &= (0, 0, 0, 0, 0, 0, 0, 0, -1), \quad q = 2 \end{aligned} \quad (5.10)$$

All other p_i are zero. Generators with $q = 0$, $q = 1$ and $q = 2$ are singlets, doublets, and triplets of $SL(2, R)$ respectively. We note that the coefficients of the expression for a general root of equation (5.3) in terms of simple roots corresponding to roots 8 and 9 are l and r respectively. The generators corresponding to the roots A and B are $R^{a_1 a_2 a_3, \alpha}$ and $R^{a_1 \dots a_5, b}$ respectively, where $\alpha, \beta = 1, 2$ indices label the vector representation of $SL(2, R)$. Root C corresponds to the generators $R^{(\alpha\beta)}$ which are none other than the generators of A_1 arising from node nine. Unlike the other representations of $A_7 \otimes A_1$ mentioned above which occur entirely within the negative level root space of E_6^{+++} with a copy in the positive root space this representation contains positive and negative levels. This is related to the fact that it belongs to the adjoint representation of $A_7 \otimes A_1$ with respect to which we are decomposing the E_6^{+++} algebra. Following the derivation of equation (5.2) and reading reference [26] one sees that in this case a negative value of r is allowed.

If $p_2 = 1$ we find the two solutions

$$\begin{aligned} \alpha_D &= (0, 0, 1, 2, 3, 2, 1, 2, 1), \quad q = 2 \\ \gamma &= (0, 0, 1, 2, 3, 2, 1, 2, 1), \quad q = 0 \end{aligned} \quad (5.11)$$

The corresponding generators are $R^{a_1 \dots a_4, (\alpha\beta)}$ and $R^{a_1 \dots a_4}$ and these have multiplicity one and zero respectively.

The fields corresponding to the above generators are two scalars ϕ and χ that belong to the coset $SL(2, R)$ with local sub-algebra $SO(2)$ associated with root C, their duals $A_{a_1 \dots a_4, (\alpha\beta)}$ which are subject to one constraint and are associated with root D, a doublet of three forms $A_{a_1 a_2 a_3, \alpha}$, whose four form field strength is self-dual and are associated with root A and finally the dual graviton $A_{a_1 \dots a_5, b}$ associated with root B. This is the content of an eight dimensional non-supersymmetric theory.

The roots of all dual generators may be written in the form

$$\begin{aligned}
\alpha_A(p_2) &= \alpha_A + p_2\gamma \\
\alpha_B(p_2) &= \alpha_B + p_2\gamma \\
\alpha_C(p_2) &= \alpha_C + p_2\gamma \\
\alpha_D(p_2) &= \alpha_D + p_2\gamma \\
\alpha_E(p_2) &= p_2\gamma
\end{aligned} \tag{5.12}$$

In Appendix A we derive all the forms arising from dimensional reduction to 4 dimensions and above.

6 G_2^{+++}

The nonlinear realisation of G_2^{+++} contains at low levels the fields of $N = 1$ five dimensional supergravity [9]. The Dynkin Diagram of G_2^{+++} is given by

$$\begin{array}{ccccccc}
& & & & 0 & & 5 \\
& & & & ||| & & \\
& & & & \widehat{|||} & & \\
& & & & ||| & & \\
0 & - & 0 & - & 0 & - & 0 \\
1 & & 2 & & 3 & & 4
\end{array}$$

The simple root vectors can be chosen to be

$$\begin{aligned}
(\alpha_5, \alpha_5) &= 2/3, \quad (\alpha_i, \alpha_i) = 2, \quad i = 1, \dots, 4, \\
(\alpha_i, \alpha_j) &= -1, \quad i, j = 1, \dots, 4, \quad |i - j| = 1 \\
(\alpha_5, \alpha_4) &= -1
\end{aligned} \tag{6.1}$$

All other scalar products are 0.

In this case, the node labelled 5 is deleted to leave an A_4 gravity subalgebra. The deleted node can be written as

$$\alpha_5 = x - \lambda_4. \tag{6.2}$$

Given $\alpha_5^2 = 2/3$ we find $x^2 = -2/15$. A general root of G_2^{+++} may be written

$$\alpha = \sum_{i=1}^4 n_i \alpha_i + l \alpha_5 = lx - \Lambda, \quad \text{where} \quad \Lambda = l \lambda_4 - \sum_i n_i \alpha_i. \tag{6.3}$$

A highest weight representation of A_4 with Dynkin indices p_i can occur if we can choose $\Lambda = \sum_i p_i \lambda_i$. Dotting this with λ_k we find

$$n_k = l(\lambda_4, \lambda_k) - \sum_j p_j (\lambda_j, \lambda_k) \tag{6.4}$$

Squaring the expression for α in (6.3) gives

$$\alpha^2 = -\frac{2}{15}l^2 + \sum_{i,j} p_i p_j (\lambda_i, \lambda_j) = \frac{1}{3}(6, 2, 0, -4, -10, -16, -22, \dots) \quad (6.5)$$

Where α^2 is constrained to take the values above as it belongs to the Kac-Moody algebra G_2^{+++} [27]. The Chevalley generator corresponding to node 5 has one A_4 index, so a level l generator, the multiple commutator for which contains the generator corresponding to node 5 l times will have l A_4 indices in total. As a result

$$l = \sum_j (5-j)p_j + 5m \quad (6.6)$$

where m denotes the number of blocks of 5 fully anti-symmetrised indices. Substituting this into (6.5) gives

$$\alpha^2 = 1/3 \left(\sum_j^4 p_j^2 (5-j)(j-2) + 2 \sum_{i<j}^4 p_i p_j (5-j)(i-2) - 4m \sum_j^4 (5-j)p_j - 10m^2 \right) \quad (6.7)$$

We now show that any generator that satisfies (6.6) possesses a root that automatically satisfies the root condition (6.4). Substituting (6.6) into (6.4) we find that

$$n_j = \sum_{k<j} p_k (j-k) + m j . \quad (6.8)$$

Note that it gives positive integer values for the root components n_k .

Dual generators are defined to be those with no blocks of 4 or 5 totally antisymmetrised indices, which may be written as

$$m = p_1 = 0 . \quad (6.9)$$

Substituting these into (6.7) gives

$$\begin{aligned} \alpha^2 &= \frac{1}{3} \left(\sum_j p_j^2 (5-j)(j-2) + 2 \sum_{i<j} p_i p_j (5-j)(i-2) \right) \\ &= \frac{1}{3}(6, 2, 0, -4, -10, -16, -22, \dots) \end{aligned} \quad (6.10)$$

Note that p_2 is absent from this formula, and the terms in the middle equation are positive.

Taking $p_2 = 0$ in equation (6.10) we get the following solutions

$$\begin{aligned} \alpha_A &= (0, 0, 0, 0, 1), p_4 = 1 \\ \alpha_B &= (0, 0, 0, 1, 2), p_3 = 1 \\ \alpha_C &= (0, 0, 0, 1, 3), p_3 = p_4 = 1 \end{aligned} \quad (6.11)$$

All other p_i are 0. The roots α_A and α_B correspond to the generators R^a and $R^{a_1 a_2}$ respectively. These give rise to the 1-form A_a and 2-form $A_{a_1 a_2}$ in the nonlinear realisation. Since the theory is in five dimensions the latter is the dual field of the former. The root α_C corresponds to the generator $R^{a_1 a_2, b}$. The corresponding field is the dual graviton. All these fields have multiplicity one. Thus at lowest levels the non-linear realisation contains gravity and a vector field as its on-shell degree of freedom [9]. This is five dimensional $N = 1$ (eight supercharges) supergravity. This theory was constructed in [29,30] and is sometimes referred to as $N = 2$ as it gives an $N = 2$ theory when dimensionally reduced.

Setting $p_2 = 1$ gives the solution

$$\gamma = (0, 0, 1, 2, 3) \quad (6.12)$$

This root vector has multiplicity 0 so does not contribute a field to the non linear realisation.

One finds that all dual generators have the roots

$$\begin{aligned} \alpha_A(p_2) &= p_2 \gamma + \alpha_A \\ \alpha_B(p_2) &= p_2 \gamma + \alpha_B \\ \alpha_C(p_2) &= p_2 \gamma + \alpha_C \\ \alpha_D(p_2) &= p_2 \gamma \end{aligned} \quad (6.13)$$

We will now find the space-filling and next to space-filling forms of the G_2^{+++} theory in five, four and three dimensions in the same way as was done for the E_8^{+++} theory in reference [16]. In five dimensions, that is the highest dimension the theory can exist, and the one considered above, we find that a three form generator appeared in equation (6.12), but this had multiplicity zero. Examining equation (6.7) we also find a solution with $p_1 = 1, m = 0$ which corresponds to a four form generator, but this also has multiplicity zero [9,31]. However, there are no scalars in this theory and so there are no fields to which the field strength of the three form can be dual. As such it is not required. That there is no four form implies that there are no deformations and so no gauged supergravities that arise from the field strength of the four form. There is a known gauging of this theory [32]. However it gauges a $U(1)$ sub-algebra of a $USp(2)$ symmetry that does not act on the bosonic fields, but only on the gravitino. Hence one does not expect the gauged theory to be associated with a deformation of the bosonic sector. To predict this gauged theory from the non-linear realisation one must first introduce the gravitino as it is only on this field that the $USp(2)$ symmetry acts.

Let us consider adding a three form and four form gauge fields and extending the known supersymmetry. Given that there is no internal symmetry in the five-dimensional G_2^{+++} non-linear realisation, the only allowed supersymmetry variation of the three form is given by $\delta A_{a_1 a_2 a_3} = i \bar{\epsilon}^i \gamma_{[a_1 a_2} \psi_{a_3]}^j \epsilon_{ij}$. However, taking the commutator of two supersymmetry transformations we find that the supersymmetry algebra does not close and so one should not introduce a three form. The same conclusion holds for the four form. This is consistent with the fact that G_2^{+++} assigns multiplicity zero to these fields.

We now consider the G_2^{+++} theory in four dimensions. This can be found by just dimensionally reducing the theory in five dimensions or deleting node four of the G_2^{+++}

Dynkin diagram and analysing the content with respect to the remaining $A_3 \otimes A_1$ algebra. The latter A_1 factor is the internal symmetry group of the four dimensional theory. The results are the same, but we will consider the former approach. Using the tables of generators of references [9,31], we find the fields; two scalars (h_5^5, A_5) which belong to the coset $SL(2, R)$ with local sub-algebra $SO(2)$, an $SL(2, R)$ quadruplet of vectors ($h_a^5, A_a, A_{a5}, A_{a5,5}$) which satisfy self-duality conditions, a triplet of two forms ($A_{a_1 a_2}, A_{a_1 a_2, 5}, A_{a_1 a_2, 5, 5}$) which are subject to one constraint and are dual to the two scalars. We also find a doublet of three forms ($A_{a_1 a_2 a_3, 5}, A_{a_1 a_2 a_3 5, 5}$) and finally a triplet four forms ($A_{a_1 a_2 a_3 a_4, 5}, A_{a_1 a_2 a_3 a_4 5, 5}, A_{a_1 a_2 a_3 a_4 5, 5, 5}$).

This theory is $N = 2$ supergravity coupled to one $N = 2$ vector multiplet which was constructed in [30] and more recently in [33]. However, the G_2^{+++} formulation is a democratic formulation in that it includes the Hodge duals of all the field strengths. As it possesses a doublet of four forms it predicts two possible deformations that is gauged supergravities. The non-linear realisation and the supersymmetry closure of this theory is under investigation and the preliminary results confirm the G_2^{+++} picture [34].

We now turn to the three dimensional G_2^{+++} theory which results from deleting node three of the G_2^{+++} Dynkin diagram leaving a G_2 internal symmetry. We will compute the forms by dimensionally reducing the five dimensional theory. We find six scalars (h_i^j, A_i, A_{ij}) which belong to the non-linear realisation G_2 with local sub-algebra $SU(3)$. There are fourteen vectors ($h_a^i, A_a, A_{ai}, A_{ai, j}, A_{aij, k}, A_{aij, kl}, A_{aij, kl, h}$). These all have multiplicity one and belong to the adjoint representation of G_2 and are dual to the scalars taking into account that they satisfy some constraints. The two forms belong to the 27 dimensional representation of G_2 and an additional singlet of G_2 (in the original version of this paper, the singlet deformation was erroneously missing, see the end of appendix B for details). In fact to find these latter forms one has to go slightly beyond the tables of references [9,31] and take into account the fact that some generators have multiplicity two. Hence, this theory possess a set of deformations parameterised by the 27 dimensional representation of G_2 and a singlet of G_2 , and a corresponding set of gauged supergravities.

In reference [35] it is pointed out that the G_2^{+++} theory does not satisfactorily encompass the known gauged supergravity of Ref. [32]. We believe that the reason for this mismatch is due to the fact that in [32] it is a $U(1)$ subgroup of the fermionic symmetry $USp(2)$ that is gauged, and given that there is no $USp(2)$ inside G_2^{+++} in five dimensions, this result is totally expected. Introducing the fermions in the non-linear realisation might lead to a solution of this problem. We will discuss this in more detail in appendix C.

7 F_4^{+++}

The Dynkin diagram of F_4^{+++} is given by

$$\begin{array}{ccccccc}
 & & & & & & 0 & 7 \\
 & & & & & & | & \\
 & & & & & & 0 & 6 \\
 & & & & & & \uparrow & \\
 0 & - & 0 & - & 0 & - & 0 & - & 0 \\
 1 & & 2 & & 3 & & 4 & & 5
 \end{array}$$

Given the Cartan matrix we may choose the root vectors to obey

$$\begin{aligned}(\alpha_i, \alpha_i) &= 2, i = 1, \dots, 5 \\(\alpha_6, \alpha_6) &= (\alpha_7, \alpha_7) = 1 \\(\alpha_i, \alpha_j) &= -1, \quad i, j = 1, \dots, 6, |i - j| = 1 \\(\alpha_6, \alpha_7) &= -1/2\end{aligned}\tag{7.1}$$

All other scalar products are 0. In the case of F_4^{+++} the node labelled 6 is deleted, leaving an A_5 gravity algebra, and an A_1 algebra. The roots of F_4^{+++} can be written as the roots of A_5 , $\alpha_i, i = 1, \dots, 5$, the root of A_1 , β , and the deleted root α_6 . The deleted root may be written as

$$\alpha_6 = x - \lambda_5 - \mu\tag{7.2}$$

where λ_5 is a fundamental weight of the A_5 subalgebra, and μ is the fundamental weight of the A_1 subalgebra. The root of the A_1 subalgebra, β , is normalised to have length 1, hence $\mu^2 = 1/4, \mu\beta = 1/2$. We find $x^2 = -1/12$.

A general root of F_4^{+++} can be written

$$\alpha = l\alpha_6 + \sum_{i=1}^5 n_i \alpha_i + r\beta = lx - \Lambda \quad \text{where} \quad \Lambda = l\lambda_5 + l\mu - \sum_i n_i \alpha_i - r\beta\tag{7.3}$$

A highest weight representation of $A_5 \otimes A_1$ with Dynkin indices p_i, q respectively can occur if we can choose $\Lambda = \sum_i p_i \lambda_i + q\mu$. Dotting this with λ_k and μ in turn gives the pair of equations

$$\begin{aligned}n_k &= l(\lambda_5, \lambda_k) - \sum_i (\lambda_i, \lambda_k) \\r &= \frac{l - q}{2}\end{aligned}\tag{7.4}$$

Squaring the expression for α in (7.3) gives

$$\alpha^2 = -\frac{1}{12}l^2 + \sum_{i,j} p_i p_j (\lambda_i, \lambda_j) + q^2/4 = 2, 1, 0, -1, \dots\tag{7.5}$$

where α^2 is constrained to take the values above.

The Chevalley generator corresponding to node 6 has one A_5 index, so a level l generator, the multiple commutator for which contains the generator corresponding to node 6 l times, will have l A_5 indices in total. The Chevalley generator corresponding to node 7 has no A_5 indices, so it does not contribute to the index total. As a result

$$l = \sum_j (6 - j)p_j + 6m\tag{7.6}$$

where m denotes the number of blocks of 6 fully antisymmetrised indices. Substituting this into (7.5) gives

$$\alpha^2 = \frac{1}{4} \sum_{j=1}^5 p_j^2 (6 - j)(j - 2) + \frac{1}{2} \sum_{i < j}^5 p_i p_j (6 - j)(i - 2) + \frac{q^2}{4} - m \sum_j (6 - j)p_j - 3m^2$$

$$= 2, 1, 0, -1, \dots \quad (7.7)$$

We now show that any generator that satisfies (7.6) automatically satisfies the root condition (7.4). Substituting (7.6) into (7.4) we find

$$n_j = \sum_{i < j} p_i(i - j) + mk, \quad j = 1, \dots, 5. \quad (7.8)$$

We note that $n_7 = r$, and $n_6 = l$.

Dual generators are defined to be those with no blocks of 5 or 6 fully anti-symmetrised A_5 indices. This may be written as

$$m = p_1 = 0. \quad (7.9)$$

Substituting these into (7.7) gives

$$\alpha^2 = \frac{1}{4} \sum_{j=1}^5 p_j^2(6-j)(j-2) + \frac{1}{2} \sum_{i < j} p_i p_j(6-j)(i-2) + \frac{q^2}{4} = 2, 1, 0, -1, \dots \quad (7.10)$$

We now find all the solutions to equation (7.10). The middle terms of this equation are positive definite with p_2 undetermined. Taking $p_2 = 0$ we get the following solutions for $\alpha^2 = 1$

$$\begin{aligned} \alpha_A &= (0, 0, 0, 0, 0, 1, 0), p_5 = 1, q = 1 \\ \alpha_B &= (0, 0, 0, 1, 2, 3, 1), p_3 = 1, q = 1 \\ \alpha_C &= (0, 0, 0, 0, 1, 2, 1), p_4 = 1, q = 0 \\ \alpha_D &= (0, 0, 0, 0, 0, 0, -1), q = 2 \end{aligned} \quad (7.11)$$

All other p_i 's are zero. The p_i s and q are the highest weight components of the $A_5 \otimes A_1$ representation. Generators with $q = 0, 1, 2$ are A_1 singlets, doublets and triplets respectively.

The generators corresponding to roots A B, C and D are $R^{a,\alpha}$, $R^{a_1 a_2 a_3, \alpha}$ and $R^{a_1 a_2}$ and $R^{(\alpha\beta)}$ respectively, where $\alpha, \beta = 1, 2$. These have multiplicity one except for $R^{a_1 a_2}$ which has multiplicity zero. The last generator is symmetric in $\alpha\beta$ and is just the triplet of generators of A_1 itself. These are at level zero with respect to n_6 and contains generators at levels $n_7 = 0 \pm 1$. The appearance of a negative level is allowed for the same reason as it did in the case of E_6^{+++} . These generators give rise to the fields expected in the non-linear realisation except for the multiplicity zero generator which leads to no field and the generator $R^{(\alpha\beta)}$ which only gives rise to two scalars ϕ and χ due to the local symmetry which being the Cartan involution invariant sub-algebra includes the $SO(2)$ part of A_1 .

The solutions for $\alpha^2 = 2$, still with $p_2 = 0$, are given by

$$\begin{aligned} \alpha_E &= (0, 0, 0, 0, 1, 2, 0), \quad p_4 = 1, q = 2 \\ \alpha_F &= (0, 0, 0, 1, 2, 4, 2), \quad p_3 = p_5 = 1, q = 0 \end{aligned} \quad (7.12)$$

These both have multiplicity one and correspond to the generators $R^{a_1 a_2, (\alpha\beta)}$ and $R^{a_1 a_2 a_3, b}$. If we take $p_2 = 1$ we find the generators

$$\begin{aligned}\alpha_G &= (0, 0, 1, 2, 3, 4, 1), \quad p_2 = 1, q = 2 \\ \gamma &= (0, 0, 1, 2, 3, 4, 2), \quad p_2 = 1, q = 0\end{aligned}\tag{7.13}$$

The second generator has multiplicity zero while the first has multiplicity one and corresponds to the generator $R^{a_1 \dots a_4, (\alpha\beta)}$.

Up to the levels considered the field content of the non-linear realisation is: two scalars ϕ, χ that belong to the coset $SO(1, 2)$, with local sub-algebra $SO(2)$, together with their duals $A_{a_1 \dots a_4, (\alpha\beta)}$ (which are subject to one constraint), two vectors $A_{a, \alpha}$ in the spinor representation and their duals $A_{a_1 a_2 a_3, \alpha}$, two self-dual two forms and one anti-self dual two form and the dual graviton. This makes up (1,0) supergravity in six dimensions ($h_a^b A_{a_1 a_2}^-$) and coupled to two vector multiplets ($A_{a, \alpha}$) and two tensor multiplets ($A_{a_1 a_2}^+$ and ϕ, χ) and well as the dual graviton $R^{a_1 a_2 a_3, b}$ [9]. The \pm on the two forms indicate their self-duality and we have not shown their $SO(1, 2)$ indices, but they combine together to form the vector representation of $SO(1, 2)$.

The roots of all dual generators may be written in the form

$$\begin{aligned}\alpha_I(p_2) &= \alpha_I + p_2 \gamma, \quad I = A, B, C, D, E, F, G \\ \alpha_H(p_2) &= p_2 \gamma\end{aligned}\tag{7.14}$$

Thus, as with all the other cases, we find the non-linear realisation contains all possible dual descriptions of the on-shell degrees of freedom of the theory.

In Appendix B we derive all the forms resulting from the F_4^{+++} non-linear realisation in 4 dimensions and above.

8 Summary of Results and Discussion

In this paper we have considered all the very extended algebras G^{+++} with the exception of the B and C series. We have deleted one, and in some cases two nodes, from the Dynkin diagram to find a preferred A_{D-1} algebra, with in two cases an additional A_1 algebra, and we have analysed the content of the algebra G^{+++} in terms of this A_{D-1} algebra. We have shown that all the generators of G^{+++} can be written with a set of A_{D-1} indices the total number of which obeys a condition that depends on the level, or levels, of the generator in question. We found that this level matching condition automatically solves, in all cases, the condition that a highest weight representation of A_{D-1} occurred amongst the root space of G^{+++} . As a result, necessary conditions for the roots of the Kac-Moody algebra G^{+++} become the condition on the length of the root squared and the level matching condition itself. Despite the fact that the level matching conditions varied from algebra to algebra the condition on the length of the root squared has a universal form given by

$$\alpha^2 = \frac{1}{(D-2)} \left(\sum_j p_j^2 (D-j)(j-2) + 2 \sum_{i < j} p_i p_j (D-j)(i-2) \right)$$

$$-\frac{4m}{(D-2)} \sum_i (D-i)p_i - \frac{2m^2 D}{(D-2)} + c \frac{q^2}{4} + \frac{4d^2}{(D-2)} = 2, \dots \quad (8.1)$$

The constant c relates to the presence of an additional A_1 algebra that survives the deletion and it is zero for all cases except for E_6^{+++} and F_4^{+++} where it is 2 and 1 respectively. The integer d is zero for all cases except for D_{D-2}^{++} where it is the difference in the two levels. For the case of a symmetric Kac-Moody algebra α^2 can only take the values $2, 0, -2, \dots$ for the non-symmetric case the possible values are given in this paper.

Consequently, the necessary condition for a representation of A_{D-1} with Dynkin indices p_j and m blocks of D totally anti-symmetrised indices to occur in the algebra G^{+++} is the condition of equation (8.1) and the level matching condition. In all cases, we have also found a formula for the G^{+++} roots in terms of their Dynkin indices p_j and m . Indeed, we can think of the set of integers $\hat{p}_{\hat{j}} = (m, p_j)$ as belonging to a lattice and equation (1.13) as containing the scalar product on this lattice. The possible generators of the Kac-Moody algebra G^{+++} correspond to points $\hat{p}_{\hat{j}}$ of the lattice that have an allowed length squared and obey the rather trivial level matching condition. This, and the physical nature of the higher level fields, suggests that it may not be totally impossible to list what are the generators of these very extended Kac-Moody algebras.

We then defined the notion of dual generators which are those that have no blocks of D and $D-1$ totally anti-symmetrised indices, that is $m = 0 = p_1$. Substituting these conditions into equation (8.1) we find it simplifies, all the terms are positive and we were able to find all possible solutions. As is apparent from equation (8.1) we always have a class of solutions with p_2 taking any positive integer value all other $p_j = 0$. It is straightforward to verify that this always satisfies the level matching condition. The solutions with $p_2 = 0$ correspond in the non-linear realisation to a set of fields that give the simplest description of the on-shell degrees of freedom of the theory together with a set of dual fields whose fields strengths are related to those of the original fields. Thus one finds in this sense a democratic formulation. The only other dual generators are all the just mentioned solutions but with the addition of the Dynkin index p_2 which can take any positive integer value. In the non-linear realisation this corresponds to adding blocks of $D-2$ totally antisymmetric indices. These encode all possible ways of writing the on-shell degrees of freedom of the theory and their presence means that the theory encodes all possible duality transformations of these on-shell degrees of freedom. With the assignment of indices to the generators given in this paper, the generators of the affine subalgebra G^+ are just given by taking the indices to only take the values $0, \dots, D-3$. In this case, all generators with blocks of $D-1$ and D indices are absent and one is left with the dual generators with the restricted index range. Indeed, the existence of the dual generators can be seen as a consequence of the affine subalgebra. One can think of the role of the dual generators as lifting the infinite number of duality relations found in two dimensions up to D dimensions.

We have explained that the G_2^{+++} theory in five dimensions does not possess three and four form fields and why this is compatible with the known gauged supergravity theory, contrary to that claimed in reference [35]. We have also computed the form fields for the G_2^{+++} theory in three and four dimensions and predicted the corresponding deformations, or gauged supergravities. A detailed analysis of all the forms arising in lower dimensions

for the cases of E_6^{+++} and F_4^{+++} is performed in two separate appendices.

Note added

The referee has asked us to comment on the occurrence of an $SL(2)$ doublet and an $SL(2)$ quadruplet of 10-forms in the E_{11} formulation of the IIB theory. Although these appeared for the first time in the tables of Ref. [9], it was in [36] that it was stressed that E_{11} predicts that the RR 10-form of IIB belongs to a quadruplet. This prediction was considered unexpected and therefore was assumed to signal a failure of the E_{11} non-linear realisation [36]. It was only later that a doublet and a quadruplet of 10-forms were shown to occur in the IIB supersymmetry algebra [12], thus giving a highly non-trivial check of the E_{11} predictions.

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A E_6^{+++} in lower dimensions

In this Appendix we determine all the forms that arise in the non-supersymmetric E_6^{+++} non-linear realisation in dimensions from 8 to 4, with the exception of the 4-forms in 4-dimensions. The scalar manifold is $Sl(2)/SO(2)$ in 8 dimensions, $Sl(2)/SO(2) \times R^+$ in 7, $[Sl(2) \times Sl(2)]/[SO(2) \times SO(2)] \times R^+$ in 6, $[Sl(3) \times Sl(3)]/[SO(3) \times SO(3)]$ in 5 and $Sl(6)/SO(6)$ in 4. We proceed using the same strategy of Ref. [16], that is we list all the 8-dimensional fields that can give rise to forms after dimensional reduction. We use the same notation as in [16], labeling each field with numbers denoting the number of antisymmetric spacetime indices in 8 dimensions. The index α is in the fundamental of $Sl(2)$. We give here the final result, where the first column denotes the highest dimension for which the corresponding fields give rise to forms after dimensional reduction:

D	fields
8	$A_3^\alpha \quad A_6^{\alpha\beta}$
7	$h_1^{-1} \quad A_{5,1} \quad A_{7,1,1}^\alpha \quad A_{8,1}^\alpha \quad A_{8,1}^{\alpha\beta\gamma}$
6	$A_{7,2}^\alpha \quad A_{8,2,1,1}$
5	$A_{6,3}^\alpha \quad A_{7,3,2} \quad A_{8,3,1}(\times 2) \quad A_{8,3,1}^{\alpha\beta}(\times 2) \quad A_{8,3,3,1}^\alpha$
4	$A_{7,4,1} \quad A_{7,4,1}^{\alpha\beta} \quad A_{7,4,4}^\alpha$

Observe that some of the fields in the list have multiplicity higher than one. All these fields, as well as the corresponding multiplicities, were listed in Ref. [31].

Performing the dimensional reduction, one obtains the results that are summarised in the following table:

D	G	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
8	$Sl(2)$			2			3		
7	$Sl(2) \otimes R^+$	1	2	2	1	1 3	2 3	2 2 4	
6	$Sl(2)^2 \otimes R^+$	2x1 1x2	2x2	2x1 1x2	1x1 3x1 1x3	2x1 1x2 2x3 3x2	1x3 2x4 2x2 2x2 4x2 3x1		
5	$Sl(3) \otimes Sl(3)$	$\overline{\mathbf{3x3}}$	3x3	8x1 1x8	$\overline{\mathbf{3x6}}$ $\mathbf{6x3}$ $\overline{\mathbf{3x3}}$	15x3 3x15 $\overline{\mathbf{6x3}}$ 3x3 3x3 3x6			
4	$Sl(6)$	20	35	70 $\overline{\mathbf{70}}$?				

The $D - 1$ -forms in the table correspond to the massive deformations that the non-linear realisation allows in dimension D .

B F_4^{+++} in lower dimensions

In this Appendix we perform for the supersymmetric F_4^{+++} case the same analysis that was carried out in the previous Appendix for the E_6^{+++} case. We determine all the forms that arise in dimensions from 6 to 4, with the exception of the 4-forms in 4 dimensions.

In six dimensions, the theory describes the bosonic sector of the gravity multiplet together with two tensor multiplets and two vector multiplets. There are two scalars, belonging to the tensor multiplets, parametrising the manifold $Sl(2)/SO(2)$. In five dimensions this corresponds to the gravity multiplet plus five vector multiplets, and the 5 scalars parametrise $Sl(3)/SO(3)$. Finally, in four dimensions the theory describes an $\mathcal{N} = 2$ gravity multiplet together with 6 vector multiplets, and the 12 scalars parametrise the manifold $Sp(6)/[SU(3) \otimes U(1)]$.

The list of all the fields that give rise to forms after dimensional reduction, together

with the highest dimension for which this occurs, is given here:

D	fields
6	$A_1^\alpha \quad A_2^{\alpha\beta} \quad A_3^\alpha \quad A_4^{\alpha\beta} \quad A_5^\alpha \quad A_5^{\alpha\beta\gamma} \quad A_6^{\alpha\beta}(\times 2) \quad A_6^{\alpha\beta\gamma\delta}$
5	$h_1^1 \quad A_{3,1} \quad A_{4,1}^\alpha \quad A_{5,1}(\times 2) \quad A_{5,1}^{\alpha\beta} \quad A_{6,1}^\alpha(\times 3) \quad A_{6,1}^{\alpha\beta\gamma}(\times 2)$ $A_{5,1,1}^\alpha \quad A_{6,1,1}^{\alpha\beta}(\times 2) \quad A_{6,1,1}^\alpha(\times 2) \quad A_{6,1,1,1}^\alpha$
4	$A_{4,2}^{\alpha\beta} \quad A_{5,2}^\alpha(\times 2) \quad A_{5,2}^{\alpha\beta\gamma} \quad A_{5,2,1} \quad A_{5,2,1}^{\alpha\beta} \quad A_{5,2,2}^\alpha$

Observe that some of the fields in the list have multiplicity higher than one. Although most of the fields were listed in the tables of [9,31] up to multiplicity 6, the fields with more than 6 indices have not appeared in the literature.

By dimensional reduction, one can then obtain all the forms that arise. The results are summarised in the Table:

D	G	A_1	A_2	A_3	A_4	A_5	A_6
6	$SL(2)$	2	3	2	3	2 4	3 5
5	$SL(3)$	6	6	8	3 15	15 24 6 3	
4	$Sp(6)$	14'	21	64	?		

The table shows in particular that the six-dimensional theory possesses massive deformations in the $\mathbf{2} \oplus \mathbf{4}$ of $SL(2)$, the five-dimensional one in the $\mathbf{3} \oplus \mathbf{15}$ of $SL(3)$, and the four-dimensional one in the $\mathbf{64}$ of $Sp(6)$.

In the original version of this paper, the forms listed in the last table of this appendix for F_4^{+++} and those listed in the last table of appendix A for E_6^{+++} , as well as the forms obtained in section 6 for G_2^{+++} , were computed using the tables of Refs. [9] and [31]. However, it turns out that for some of the higher rank forms these tables are not sufficiently exhaustive to contain all the required fields. We have used the program SimpLie [19], available on <http://strings.fmns.rug.nl/SimpLie/>, to compute the extra fields and have subsequently modified the tables and corrected section 6. We thank the referee for drawing this to our attention.

C G_2^{+++} and minimal gauged supergravity in five dimensions

The minimal supergravity multiplet in five dimensions contains the metric, a $U(1)$ vector A_μ and a $USp(2)$ doublet of gravitini ψ_μ^i satisfying symplectic Majorana conditions. The $USp(2)$ symmetry only acts on the gravitini, and in [32] it was shown that this theory admits a gauging of a $U(1)$ subgroup of the $USp(2)$ symmetry. This gauging is thus different from the ones that occur in maximal supergravity because it arises from a symmetry that only acts on the fermions.

The field equation for the vector at lowest order in the fermions, in both the massless and the gauged case, has the form

$$D_\mu F^\mu{}_\nu = -\frac{1}{2\sqrt{6}}\sqrt{-g}\epsilon_{\mu\mu_1\dots\mu_4}F^{\mu_1\mu_2}F^{\mu_3\mu_4} \quad (C.1)$$

because of the presence of a Chern-Simons term $A \wedge F \wedge F$ in the lagrangian, where $F_{\mu\nu}$ is the field strength of A_μ . In the gauged theory the lagrangian contains a minimal coupling g of the gravitino to the vector, as well as a mass term g for the gravitino and a cosmological constant $-g^2$. The lagrangian reads, up to quartic order fermi terms,

$$\begin{aligned} \mathcal{L} = & \sqrt{-g}\left[-\frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_{\rho i} + \frac{1}{2}g\bar{\psi}_\mu\gamma^{\mu\nu\rho}A_\nu\delta_{ij}\psi_\rho^j - i\frac{\sqrt{6}}{4}g\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu^j\delta_{ij}\right. \\ & \left.+ 4g^2 - \frac{3}{8\sqrt{6}}i\bar{\psi}_\mu(\gamma^{\mu\nu\rho\sigma} + 2g^{\mu\nu}g^{\rho\sigma})\psi_{\sigma i}F_{\nu\rho}\right] + \frac{1}{6\sqrt{6}}\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} \quad . \end{aligned} \quad (C.2)$$

The fermionic terms that arise in the gauged theory, that is the order g terms in eq. (C.2), are proportional to δ_{ij} , which breaks $USp(2)$ explicitly.

The authors of [35] extended the algebra of this model introducing a 2-form dual to the vector, as well as 3-forms and 4-forms that are both non-propagating. In five dimensions a 3-form is dual to a scalar but in [35] the 3-forms are introduced regardless the fact that there are no scalars in the model, and therefore their field strength is required to vanish identically. The authors originally showed that the supersymmetry algebra allows the introduction of a 3-form and a 4-form whose field strength is dual to the coupling constant g in the gauged theory. In the first version of this paper, we pointed out that this would have led to an explicit symmetry breaking of $USp(2)$ in the algebra in the ungauged theory, and an explicit breaking of the R symmetry is inconsistent with supersymmetry. The authors then revised their paper showing that both the 3-form and the 4-form are actually contained in a $USp(2)$ triplet, and inserting a footnote to acknowledge our contribution.

The G_2^{+++} non-linear realisation in five dimensions describes the bosonic sector of this model, and in particular it contains a vector and its dual 2-form [9]. No 3-forms and 4-forms are present, and the authors of [35] pointed out that the mass parameter of the gauged supergravity of [32] can not be obtained as the dual of a 4-form arising in the G_2^{+++} non-linear realisation, because no such forms are present in this theory in 5 dimensions.

We now summarise the results of ref. [35]. In the ungauged theory, the supersymmetry algebra closes on the 2-form $B_{\mu\nu}$ as expected from G_2^{+++} , imposing that its field strength

$$G_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \sqrt{6}A_{[\mu}F_{\nu\rho]} \quad (C.3)$$

is dual to the field strength of the vector $F_{\mu\nu}$. Taking the curl of the duality relation

$$G_{\mu\nu\rho} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma\tau}F^{\sigma\tau} \quad (C.4)$$

one recovers the vector field equation (C.1). The $USp(2)$ triplet of 3-forms $C_{\mu\nu\rho}^{ij}$ not found in G_2^{+++} transforms under supersymmetry as

$$\delta C_{\mu\nu\rho}^{ij} = i\bar{\epsilon}^{(i}\gamma_{[\mu\nu}\psi_{\rho]}^{j)} \quad (C.5)$$

and the commutator of two supersymmetry transformations closes at lowest order in the fermions imposing that the field strength vanishes identically,

$$H_{\mu\nu\rho\sigma}^{ij} = \partial_{[\mu} C_{\nu\rho\sigma]}^{ij} = 0 \quad . \quad (C.6)$$

The 4-forms $D_{\mu\nu\rho\sigma}^{ij}$ transform under supersymmetry as

$$\delta D_{\mu\nu\rho\sigma}^{ij} = \bar{\epsilon}^{(i} \gamma_{[\mu\nu\rho} \psi_{\sigma]}^{j)} - i\sqrt{6} A_{[\mu} \bar{\epsilon}^{(i} \gamma_{\nu\rho} \psi_{\sigma]}^{j)} \quad (C.7)$$

and their field strength vanishes identically, *i.e.*

$$L_{\mu\nu\rho\sigma\tau}^{ij} = \partial_{[\mu} D_{\nu\rho\sigma\tau]}^{ij} = 0 \quad . \quad (C.8)$$

The supersymmetry transformation of the gravitino has the form [32]

$$\delta\psi_{\mu}^i = D_{\mu}\epsilon^i + i\frac{1}{4\sqrt{6}}(\gamma_{\mu}^{\nu\rho} - 4\delta_{\mu}^{\nu}\gamma^{\rho})F_{\nu\rho}\epsilon^i \quad (C.9)$$

and at lowest order in the fermions one has to only consider the variations of the gravitino in eqs. (C.5) and (C.7). This only produces gauge transformations, while the general coordinate transformations are generated from the fact that the field strengths of eq. (C.6) and (C.8) vanish identically. Thus, the fact that the field strengths vanish identically implies that these fields are pure gauge, and thus supersymmetry closes on these fields in a rather trivial way, as the authors point out [35].

The procedures used in [35] and the ones used in the E_{11} formulation of maximal supergravities are different for a number of reasons. First of all, in Ref. [35] the introduction of 3-forms and 4-forms is trivial in the sense that their field strengths of the 3-form and the 4-form do not contain any lower rank field. This is in contrast with the E_{11} case, in which the field strengths of higher rank contain the fields of lower rank. Secondly, while in E_{11} the $D - 2$ forms are dual to scalars, and their field strengths are not identically zero, in the model of [35] there is no real democracy because there are no scalars and this makes the introduction of 3-forms somehow artificial. Finally, the 3-forms and 4-forms are triplets of the fermionic global symmetry $USp(2)$ and thus are not related to the bosonic symmetry that arises in the non-linear realisation (which is absent in this five-dimensional case). The closure of the supersymmetry algebra does rely on properties of the γ matrices. However, the 3-forms and 4-forms do not couple to the other fields and thus the closure is a rather trivial consequence of the γ matrix identities. The democratic formulation of the supersymmetry algebra of maximal five dimensional supergravity theory has recently been described in [37]. In that case the 3-forms are in the adjoint of E_6 , and are dual to the scalars that realise non-linearly E_6 with local subalgebra $USp(8)$. One can see from that result that one needs the same γ matrix identities to cancel the $F_{\mu\nu}$ terms in the supersymmetry commutator on the 3-forms, and thus putting to zero the scalars and the field strengths of the 3-forms is from this point of view a singular limit, in which 3-forms arise because one is using only a part of the constraints that come from the algebra when the scalars are present.

We now study the gauged theory of [35]. The supersymmetry variation of the gravitino is modified by the addition of a term of the form [32]

$$\delta'\psi_\mu^i = -gA_\mu\delta^{ij}\epsilon_j - i\frac{1}{\sqrt{6}}g\gamma_\mu\delta^{ij}\epsilon_j \quad . \quad (C.10)$$

This term does not affect the supersymmetry commutator on the 3-forms, while it does affect the commutator on the 4-forms, which now closes provided that the duality relation

$$L_{\mu\nu\rho\sigma\tau}^{ij} \sim g\epsilon_{\mu\nu\rho\sigma\tau}\delta^{ij} \quad (C.11)$$

holds, where $L_{\mu\nu\rho\sigma\tau}^{ij}$ is the gauge invariant field strength defined in (C.8). The supersymmetry algebra on the 2-form implies that the gauge invariant field strength for the 2-form is now

$$G_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} - \sqrt{6}A_{[\mu}F_{\nu\rho]} - 3g\delta_{ij}C_{\mu\nu\rho}^{ij} \quad . \quad (C.12)$$

Thus, the 2-form transforms with respect to the gauge parameter of the 3-form, which therefore can be used to gauge away the 2-form completely. Secondly, the field strengths of the 4-forms are dual to the coupling constant g . We observe that the field strengths of the 3-forms and the 4-forms are not modified with respect to the massless case, and in particular the 3-forms are pure gauge quantities also in the gauged case. Again, like in the massless case there is no real democracy, and in particular one can introduce the 4-forms dual to the mass parameters without needing to introduce the 2-form and the 3-forms. This is in contrast with what happens in the E_{11} non-linear realisation. Finally, if one believes that $D - 1$ forms are responsible for gauged supergravities, the existence of a triplet of 4-forms as proposed in [35] would predict three such theories. However, only one such theory exists. This is again in contrast with what happens in the E_{11} case, in which the number of deformations and the number of $D - 1$ forms coincide in any dimension.

We believe that there are exceptional cases for which the gauged theory is not accounted for by the bosonic degrees of freedom in G^{+++} but by the fermionic partners as treated from the view point of the non-linear realisation. This particular model is singular in the sense that it has no scalars and has a global symmetry that only acts on the fermions.

As a toy model showing that the analysis of higher rank forms becomes more singular in cases with fewer supersymmetry and no scalars, we consider the very well known theory of pure minimal supergravity in 4 dimensions. In this case the supergravity multiplet only contains the metric and a Majorana gravitino, and one can introduce a negative cosmological constant $-g^2$ and a mass term for the gravitino, that schematically correspond to the appearance in the action of the terms

$$\text{dete}(g\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + g^2) \quad (C.13)$$

and supersymmetry is provided by the fact that the gravitino transforms as

$$\delta\psi_\mu = D_\mu\epsilon + g\gamma_\mu\epsilon \quad . \quad (C.14)$$

We first want to consider the introduction of 2-forms. These would be the equivalent of 3-forms in five dimensions. In the massless theory the supersymmetry algebra closes trivially on a 2-form $B_{\mu\nu}$ whose supersymmetry transformation is

$$\delta B_{\mu\nu} = \bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} \quad (C.15)$$

if one imposes

$$\partial_{[\mu}B_{\nu\rho]} = 0 \quad . \quad (C.16)$$

In the massive theory this field can no longer be introduced because the supersymmetry commutator produces a term of the form $g\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2$ which can not be interpreted as a gauge transformation of any kind.

We now move to the 3-forms. One can introduce in both the massless and the massive theory a 3-form $C_{\mu\nu\rho}$ whose supersymmetry variation is

$$\delta C_{\mu\nu\rho} = \bar{\epsilon}\gamma_{[\mu\nu}\gamma_5\psi_{\rho]} \quad (C.17)$$

if one imposes

$$\partial_{[\mu}C_{\nu\rho\sigma]} = g\epsilon_{\mu\nu\rho\sigma} \quad . \quad (C.18)$$

This form is therefore dual to the mass parameter g . In addition to this, one can also introduce a 3-form $C'_{\mu\nu\rho}$ whose supersymmetry variation is

$$\delta C'_{\mu\nu\rho} = \bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} \quad (C.19)$$

if one imposes

$$\partial_{[\mu}C'_{\nu\rho\sigma]} = 0 \quad . \quad (C.20)$$

This additional 3-form has therefore vanishing field strength in both the massless and the massive theory. Therefore, in this model one can introduce a 2-form only if $g = 0$, while for any value of the coupling constant one can introduce two 3-forms, one of them being dual to g . While it seems that for any massive supersymmetric theory one can introduce a $D - 1$ form whose field strength is dual to the mass deformation parameter, we think that this model reveals the singular nature of these manipulations in theories that are not enough constrained by supersymmetry.

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